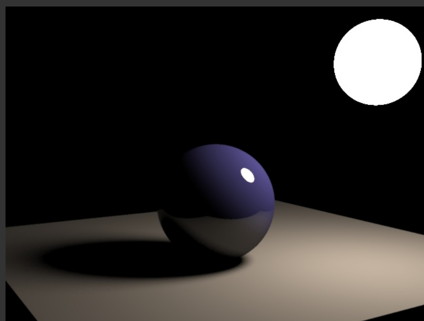
CSE168: Rendering Algorithms
Monte Carlo Ray Tracing



Henrik Wann Jensen

henrik@cs.ucsd.edu

Todays Menu

- What is Monte Carlo
- Probability
- Monte Carlo integration
- Monte Carlo sampling
- Area lights
- Glossy materials

Monte Carlo History

- Use Random numbers (therefore the name)
- Early use in neutron transport
- von Neumann
- Metropolis

Monte Carlo Algorithms

The good:

- Flexible
- Easy to implement
- Often easy to understand
- Robust in complex domains
- Efficient for high-dimensional problems

Monte Carlo Algorithms

The bad:

- Noisy (variance)
- Slow convergence (requires many samples)

Random Variables

Random variable: X

Probability distribution function (PDF): $p(x)$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

Random Variables

Cumulative probability distribution function: $P(x)$

$$P(x) = \int_{-\infty}^{x} p(\mu)\, d\mu$$

$$P(a \le X \le b) = \int_{a}^{b} p(x)\, dx$$

Expected Value

Expected value: $E\{X\}$

$$E\{X\} = \int x p(x)\, dx$$

$$E\{X + Y\} = E\{X\} + E\{Y\}$$

Expected Value

$$E\{X\} \approx \frac{1}{N} \sum_{i=1}^{N} x_i$$

where x_i are distributed according to $p(x)$

$$\text{Probability} \left[E\{X\} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i \right] = 1$$

Example: Random Numbers

If ξ is a uniformly distributed random number between 0 and 1 then the PDF $p(x)$ is:

$$p(x) = \begin{cases} 1 & \text{if } 0 \le \xi \le 1 \\ 0 & \text{else} \end{cases}$$

$$E\{X\} = \int_{-\infty}^{\infty} x\, p(x)\, dx = 0.5$$

$$P(x < 0.7) = \int_{-\infty}^{0.7} p(x)\, dx = \int_{0}^{0.7} 1\, dx = 0.7$$

Monte Carlo Integration

We want to compute the value I of an integral

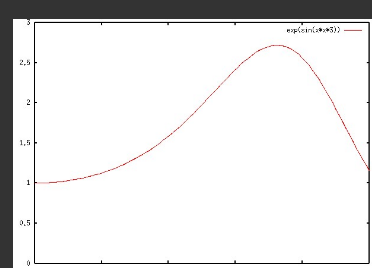
$$I = \int_{0}^{1} f(x)\, dx$$

$$E\{f(X)\} = \int f(x) p(x)\, dx \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

$$I = \int f(x)\, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

Monte Carlo Integration

$$f(x) = e^{\sin(3x^2)}$$



Monte Carlo Integration

```
double integrate( int N )
{
  double x, sum=0.0;
  for (int i=0; i<N; i++) {
    x = drand48();
    sum += exp(sin(3*x*x));
  }
  return sum / double(N);
}
```


Monte Carlo Integration

$$f(x) = e^{\sin(3x^2)}$$

N	I
1	2.75039
10	1.9893
100	1.79139
1000	1.75146
10000	1.77313
100000	1.77862

Monte Carlo Integration

$$\begin{aligned} V\{f(X)\} &= E\{f(X)^2\} - [E\{f(X)\}]^2 \\ &= V\left\{\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right\} \\ &= \frac{1}{N^2}\sum_{i=1}^{N} V\{f(x_i)\} \\ &= \frac{1}{N} V\{f(x_i)\} \end{aligned}$$

Monte Carlo Convergence

Convergence: $\sigma \propto \frac{1}{\sqrt{N}}$

- Slow :-(
- Independent of the dimension

Monte Carlo Integration

Improvements:

- Stratified sampling
- Importance sampling

Stratified Sampling

$$f(x) = e^{\sin(3x^2)}$$

N	I
1	2.70457
10	1.72858
100	1.77925
1000	1.77606
10000	1.77610
100000	1.77610

Convergence: $\sigma \propto \frac{1}{N}$

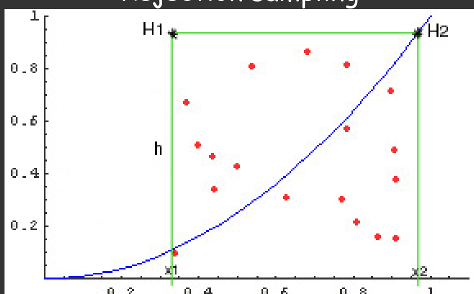
A Visual Break

Monte Carlo Sampling

- Rejection sampling
- PDF based sampling

Rejection Sampling

Rejection sampling



PDF Based Sampling

Sampling a continuous distribution

Cumulative probability distribution function

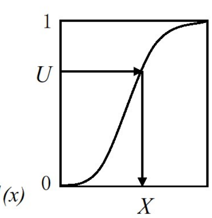
$P(x) = \Pr(X < x)$

Construction of samples

Solve for $X = P^{-1}(U)$

Must know:

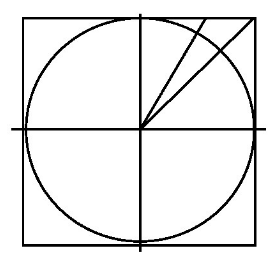
1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$



Monte Carlo Sampling

- Integrate over domain
  - Disc
  - Triangle
  - Hemisphere
- Sample using ray tracing

Rejection Sampling a Disc



```
do {
   X=1-2*U1
   Y=1-2*U2
while( X²+ Y² >1 )
```

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

Direct Sampling of a Disc

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{\pi} r\,dr\,d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

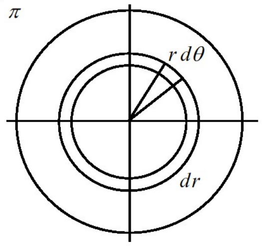
$$p(r,\theta) = p(r)p(\theta)$$

$$p(\theta) = \frac{1}{2\pi} \qquad \theta = 2\pi U_1$$

$$P(\theta) = \frac{1}{2\pi}\theta$$

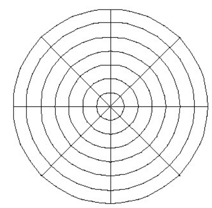
$$p(r) = 2r \qquad r = \sqrt{U_2}$$

$$P(r) = r^2$$



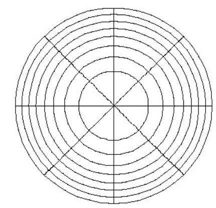
Sampling of a Disc

WRONG ≠ Equi-Areal



$$\theta = 2\pi U_1$$
$$r = U_2$$

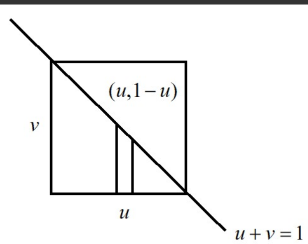
RIGHT = Equi-Areal



$$\theta = 2\pi U_1$$
$$r = \sqrt{U_2}$$

Sampling a Triangle

$$u \ge 0$$
$$v \ge 0$$
$$u + v \le 1$$



$$A = \int_0^1 \int_0^{1-u} dv\,du = \int_0^1 (1-u)\,du = -\frac{(1-u)^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

Sampling a Triangle

Here u and v are not independent! $p(u,v) = 2$

Conditional probability

$$p(u) \equiv \int p(u,v)\,dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(u)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\,du = (1-u_0)^2 \qquad u_0 = 1 - \sqrt{U_1}$$

$$p(v \mid u) = \frac{1}{(1-u)} \qquad v_0 = \sqrt{U_1}U_2$$

$$P(v_0 \mid u_0) = \int_0^{v_0} p(v \mid u_0)\,dv = \int_0^{v_0} \frac{1}{(1-u_0)}dv = \frac{v_0}{(1-u_0)}$$

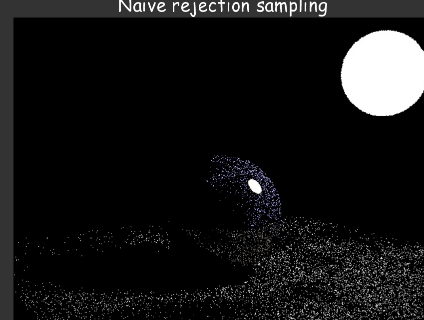
Diffuse Lighting

$$L(x,\vec{\omega}) = \frac{R_d}{\pi}\int_{2\pi} L_i(x,\vec{\omega}')(\vec{n}\cdot\vec{\omega}')\,d\vec{\omega}'$$

- Accounts for all direct light on a diffuse surface
- Use Monte Carlo integration

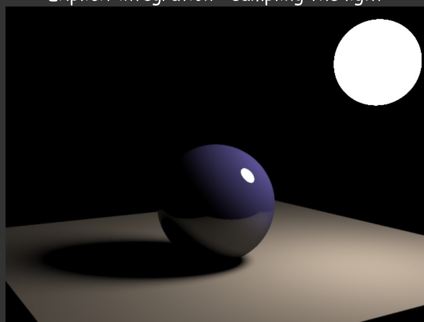
Sphere Light

Naive rejection sampling



Sphere Light

Explicit integration - sampling the light



Sphere Light

Procedure:

- Find disc oriented towards x
- Create samples in disc
- Trace ray to sample visibility

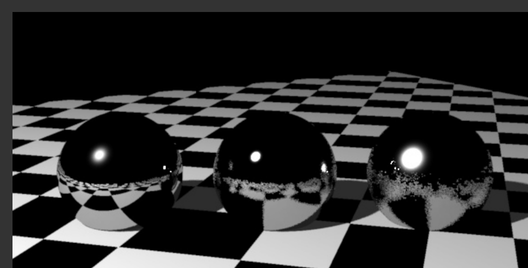
Glossy Phong

$$p(\theta, \phi) = \frac{n+1}{2\pi} \cos^n \theta$$

$$P(\theta, \phi) = \int_0^{\phi} \int_0^{\theta} p(\theta', \phi') \sin\theta' \, d\theta' \, d\phi'$$

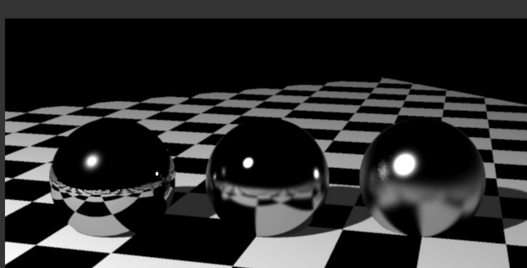
$$(\theta, \phi) = (\ acos((1-u)^{1/(n+1)}),\ 2\pi v)$$

Glossy Spheres



1 sample

Glossy Spheres



256 samples

Next time

- Global illumination
  - ⋆ The rendering equation
  - ⋆ Light transport
  - ⋆ More Monte Carlo...
  - ⋆ Finite Element Radiosity